Recursive Time-Frequency Reassignment

Geir Kjetil Nilsen

***Abstract*—A fast algorithm for creating time-frequency representations based on a special case of the short-time Fourier transform (STFT) is presented. The algorithm is extended with the method known as time-frequency reassignment. This approach makes time-frequency reassignment well suited for real-time implementations.**

***Index Terms*—STFT, short-time Fourier transform, spectrogram, recursion, time-frequency analysis, reassignment, real-time**

I. Introduction

THE idea of calculating the STFT in terms of recursions is not new [1], [2], [3], [4], [5], [6]. Chen et al. [4] was the first to utilize a causal, infinite window function to compute the STFT recursively, although Friedlander et al. [7] several years earlier used the same window function with the Gabor transform for transient detection. Žnidar [2] developed a perfect reconstruction filter bank based on the same type of underlying filters, but did not at that point characterize this as a STFT with an infinite window. Tomažic and Žnidar [1], [3] together developed a fast recursive STFT algorithm based on the same concept. The main difference from [4] is that their algorithm has support for optional redundancy in the time-dimension. It was also shown that as the number of repeated poles on the window increases, the Gabor uncertainty approached the optimal Gaussian uncertainty. Unser [8] also employed these windows as wavelets. Amin and Feng [5] at about the same time generalized the STFT with arbitrary infinite windows using cascading (IIR) filter structures. Richard and Lengellé [6] developed recursive structures allowing to recursively calculate the STFT for a whole a class of windows including the most common finite acausal windows such as the Hamming, Hanning and Blackman.

As will be seen, the Laplace transform of the infinite causal window used in this paper has a multipole, and the number of poles offers a computational cost to time-frequency resolution tradeoff. Moreover will the discrete implementation be based on recursive expressions which are computationally efficient and easy to implement. Further the readability of the of the resulting spectrogram can be enhanced by introducing the method known as time-frequency reassignment. The extension to reassignment is shown to be of equally low computational cost.

The pioneers of the reassignment method, Kodera et al. [9], [10] were influenced by the complex energy density theory of Rihaczek [11]. It was demonstrated that the readability of the spectrogram can be improved by reassigning its points to new coordinates. The new coordinates were found to be related to the partial derivatives of the STFT phase. Kodera et. al did, nevertheless, only use approximative finite differences in their implementation. This also required use of the unreliable operation known as phase-unwrapping [12]. Nelson [13] invented his approximative cross-spectral method making phase-unwrapping superfluous. Auger and Flandrin [14] generalized the reassignment method to all the members of Cohen's class [15], and it was shown that the reassignment coordinates for the STFT spectrogram can be computed exactly by only two extra STFTs using two modified windows. Richard and Lengellé [6] also extended their recursive STFT structures with the reassignment method. The reassignment method has been used and discussed in several areas of research [13], [16], [17], [18].

This paper is organized as follows: in Section II a special case of the STFT yielding a recursive linear system is presented using a new formulation. Moreover is it shown how the system can be discretized to yield digital filter coefficients. Section III shows that the time-frequency reassignment coordinates can be computed efficiently by two simple modifications of the systems transfer function. Further the computational cost of the algorithm is briefly discussed, and a recursively reassigned spectrogram is shown to prove its correctness. Section IV sums up the paper.

II. The STFT Special Case

The STFT is the continuous version of the Gabor transform [19]. It can be regarded as a set of linear convolutions,

$$y(t,\omega) = e^{i\omega t}\int_{-\infty}^{\infty} x(\tau)\mathcal{W}(t-\tau)e^{-i\omega\tau}d\tau, \tag{1}$$

between the input signal $x(\cdot)$ and the impulse responses given by

$$h(t,\omega) = \mathcal{W}(t)e^{i\omega t}, \tag{2}$$

where $\mathcal{W}(\cdot)$ is the window. The spectrogram is defined as the modulus squared of (1).

By a careful substitution of $\mathcal{W}(\cdot)$, (1) can become a solution to an ordinary differential equation. The differential will act on the time dimension, and in the discrete domain be equivalent to a recursion.

If the window is defined

$$\mathcal{W}_k(t) = \frac{\sigma^k t^{k-1}}{(k-1)!}e^{-\sigma t}u(t), \tag{3}$$

with $u(\cdot)$ being the Heaviside step function, and $k \geq 2$ to be explained subsequently, then the impulse response (2) becomes

$$h_k(t,\omega) = \mathcal{W}_k(t)e^{i\omega t}. \tag{4}$$

The transfer function of (4) is given by

$$H_k(s) = \frac{\sigma^k}{(s-p)^k}, \tag{5}$$

G. K. Nilsen is at the Department of Computer Science, University of Bergen, Norway, e-mail: geni@bjorge.no

where the multipole p is a complex constant defined by

$$p = -\sigma + i\omega. \tag{6}$$

It is evident that (5) is a rational function with one pole raised to the kth power. This means that (4) consists of k repeated convolutions of first order impulse responses.

Using (4), the corresponding set of linear convolutions becomes

$$y_k(t,\omega) = \int_{-\infty}^{\infty} x(\tau) h_k(t-\tau,\omega) d\tau. \tag{7}$$

The STFT in (1) is equivalent to (7) using the special window in (4), but (7) is also a solution to the kth order ordinary differential equation

$$\sum_{n=0}^{k} \binom{k}{n} (-\sigma + i\omega)^n \frac{\partial^{k-n}}{\partial t^{k-n}} y_k(t,\omega) = \sigma^k x(t), \tag{8}$$

with all initial conditions equal to zero. Notice that (7) is equivalent (8), and that the discretized version of (8) can be computed efficiently in terms of recursions.

A. Time-Frequency Resolution

The classical definition of bandwidth cannot be applied, because the second moment of $|H_k(s)|^2|_{s=i\nu}$ does not converge. Reasonable definitions for the bandwidth and the duration of (5) and (4), respectively, were proposed by [1]. The definition of the bandwidth is the duration of a rectangular window with height equal to $\max|H_k(s)|^2\,|_{s=i\nu}$ and energy equal to the energy of $H_k(s)\,|_{s=i\nu}$, i.e.

$$\Delta_\omega = \frac{1}{\max|H_k(i\nu)|^2} \int_{-\infty}^{\infty} |H_k(i\nu)|^2 d\nu. \tag{9}$$

The bandwidth of (5) using this definition gives

$$\Delta_\omega = \frac{2\pi\sigma[2(k-1)]!}{[(k-1)!]^2 2^{2k-1}}. \tag{10}$$

For the duration, an equivalent definition is used on (4). The duration using this definition is found to be

$$\Delta_t = \frac{[2(k-1)]!}{\sigma\,(k-1)^{2(k-1)}\, e^{-2(k-1)} 2^{2k-1}}. \tag{11}$$

The time-frequency resolution is inversely proportional to the uncertainty $\Delta_t\Delta_\omega$. Using $k=2$ yields $\Delta_t\Delta\omega = 2.901$, while the same definitions applied on the optimal Gaussian window would yield $\Delta_t\Delta\omega = \pi$. As the number of repeated poles (k) in (5) increases, the time-frequency resolution approaches the optimum. This is a direct consequence of the central limit theorem, readily understood through the repeated convolutions in (4). The convergence is fast enough to yield $\Delta_t\Delta_\omega = 3.077$ when $k=5$.

B. Discretization

Using the impulse invariant discretization method [20], [21], the inverse Laplace transform is applied to equation (5), and the resulting impulse response (4) can be sampled such that the sampling period T is sufficiently small in Nyquist sense. Using the z-transform relation

$$\mathcal{Z}\{n^k\alpha^n\} = -z\frac{\partial}{\partial z}\mathcal{Z}\{n^{k-1}\alpha^n\}, \tag{12}$$

and defining the generally complex constant

$$\alpha = e^{pT}, \tag{13}$$

the resulting digitized transfer function is induced to be given by

$$H_k(z) = \frac{\sigma^k T^k}{(k-1)!} \frac{\sum_{j=1}^{k} A_{k-1,k-j}(\alpha z^{-1})^{j-1}}{(1-\alpha z^{-1})^k}, \tag{14}$$

with $A_{k,n}$ representing the Eulerian numbers given by

$$A_{k,n} = \sum_{j=0}^{n} (-1)^j \binom{k+1}{j} (n+1-j)^k. \tag{15}$$

Table (I) lists the coefficients for orders $k=2$ to $k=5$. Increasing k also means increasing the number of filter coefficients, so the parameter k gives a tradeoff between computational cost and time-frequency resolution. The implementation can now be done by using these coefficients with the standard difference equation[1]

$$\begin{aligned} y_k(nT, 2\pi f\Omega) \approx & \sum_{j=0}^{k-1} b_j x(nT - jT) \\ & - \sum_{j=1}^{k} a_j y_k(nT - jT, 2\pi f\Omega), \end{aligned} \tag{16}$$

with Ω denoting the frequency spacing and with $n = 0, 1, \ldots, N-1$ and $f = 0, 1, \ldots, \mathcal{F}-1$. Note that the filter coefficients a_j and b_j must correspond to the order k, and that they actually become functions of the frequency variable f.

The real part of the multipole in (3) controls the decay rate of the exponential. This parameter can therefore be used to balance the tradeoff between time and frequency resolution. By setting $\Delta_\omega/\Omega = \Delta_t/T$, and solving for σ, one finds the σ which makes the bandwidth (in number of samples) equal to the duration (in number of samples). This has the effect of balancing the time-frequency resolution in number of samples so that the result can easily be visualized. It is given by

$$\sigma = \frac{\sqrt{\Omega}(k-1)!}{\sqrt{2\pi T}(k-1)^{k-1}e^{1-k}}. \tag{17}$$

Note that the balancing parameter σ is cancelled in $\Delta_t\Delta_\omega$, so it has nothing to do with the time-frequency resolution (or uncertainty). In the classical STFT, the number of samples in the window is used to balance time and frequency resolution tradeoff. However, the window length in the special case is implicitly infinite, so the time-frequency resolution balance can only be influenced by the parameter σ. Also note that quite simply, one can as well get constant Q analysis by choosing σ such that ω is proportional to Δ_ω.

[1]For instance available through the MATLAB function filter().

k	b_n	a_n
2	$b_0 = 0, b_1 = \sigma^2 T^2 \alpha$	$a_0 = 1, a_1 = -2\alpha, a_2 = \alpha^2$
3	$b_0 = 0, b_1 = \sigma^3 T^3 \frac{\alpha}{2}$, $b_2 = \sigma^3 T^3 \frac{\alpha^2}{2}$	$a_0 = 1, a_1 = -3\alpha$, $a_2 = 3\alpha^2, a_3 = -\alpha^3$
4	$b_0 = 0, b_1 = \sigma^4 T^4 \frac{\alpha}{6}$, $b_2 = \sigma^4 T^4 \frac{2\alpha^2}{3}$, $b_3 = \sigma^4 T^4 \frac{\alpha^3}{6}$	$a_0 = 1, a_1 = -4\alpha$, $a_2 = 6\alpha^2, a_3 = -4\alpha^3$, $a_4 = \alpha^4$
5	$b_0 = 0, b_1 = \sigma^5 T^5 \frac{\alpha}{24}$, $b_2 = \sigma^5 T^5 \frac{11\alpha^2}{24}$, $b_3 = \sigma^5 T^5 \frac{11\alpha^3}{24}$, $b_4 = \sigma^5 T^5 \frac{\alpha^4}{24}$	$a_0 = 1, a_1 = -5\alpha$, $a_2 = 10\alpha^2, a_3 = -10\alpha^3$, $a_4 = 5\alpha^4, a_5 = -\alpha^5$

TABLE I
SUMMARY OF FILTER COEFFICIENTS FOR ORDERS $k = 2...5$

k	b_n	a_n
2	$b_0 = \sigma^2 T$, $b_1 = -\sigma^2 T \alpha (1 - pT)$,	$a_0 = 1, a_1 = -2\alpha$, $a_2 = \alpha^2$
3	$b_0 = 0, b_1 = \sigma^3 T^2 \frac{\alpha}{2}(2 + pT)$, $b_2 = -\sigma^3 T^2 \frac{\alpha^2}{2}(2 - pT)$	$a_0 = 1, a_1 = -3\alpha$, $a_2 = 3\alpha^2, a_3 = -\alpha^3$
4	$b_0 = 0, b_1 = \sigma^4 T^3 \frac{\alpha}{6}(3 + pT)$, $b_2 = \sigma^4 T^3 \frac{2\alpha^2}{3} pT$, $b_3 = -\sigma^4 T^3 \frac{\alpha^3}{6}(3 - pT)$	$a_0 = 1, a_1 = -4\alpha$, $a_2 = 6\alpha^2, a_3 = -4\alpha^3$, $a_4 = \alpha^4$

TABLE II
SUMMARY OF FREQUENCY REASSIGNMENT FILTER COEFFICIENTS FOR ORDERS $k = 2...4$

III. RECURSIVE REASSIGNMENT

In analogy with Auger and Flandrin's method [14], this section shows that if the window in (3) is used, the reassignment coordinates can be computed recursively as well.

Since the coordinates for the reassignment are related to the partial derivatives of the STFT phase [9], the frequency reassignment coordinate can be found by

$$\begin{aligned}\hat{\omega}(t,\omega) &= \frac{\partial}{\partial t}\arg\left[y_k(t,\omega)\right] \\ &= \frac{\partial}{\partial t}\mathrm{Im}\left[\log y_k(t,\omega)\right] \\ &= \mathrm{Im}\left[\frac{\partial}{\partial t}\log y_k(t,\omega)\right] \\ &= \mathrm{Im}\left[\frac{\frac{\partial}{\partial t}y_k(t,\omega)}{y_k(t,\omega)}\right] \\ &= \mathrm{Im}\left[\frac{y_{\mathcal{D}k}(t,\omega)}{y_k(t,\omega)}\right], \end{aligned} \tag{18}$$

and likewise the time reassignment coordinate by

$$\begin{aligned}\hat{t}(t,\omega) &= t - \frac{\partial}{\partial \omega}\arg\left[y_k(t,\omega)\right] \\ &= t - \mathrm{Im}\left[\frac{\frac{\partial}{\partial \omega}y_k(t,\omega)}{y_k(t,\omega)}\right] \\ &= t - \mathrm{Im}\left[\frac{i y_{\mathcal{T}k}(t,\omega)}{y_k(t,\omega)}\right] \\ &= t - \mathrm{Re}\left[\frac{y_{\mathcal{T}k}(t,\omega)}{y_k(t,\omega)}\right], \end{aligned} \tag{19}$$

where $y_k(\cdot)$ represents the output from the kth order recursive system described by (5) or (7). The differentiation of $y_k(\cdot)$ with respect to time propagates into the transfer function of the recursive system as a multiplication by s in the Laplace domain. The new frequency coordinate in (18) can therefore be found by applying the frequency reassignment filter

$$H_{\mathcal{D}k}(s) = sH_k(s), \tag{20}$$

to the input signal $x(\cdot)$ to get $y_{\mathcal{D}k}(\cdot)$ for use in the numerator of (18). The differentiation of $y_k(\cdot)$ with respect to frequency also yields a simple relation. The new time coordinate (19) can be found by applying the time reassignment filter

$$H_{\mathcal{T}k}(s) = -\frac{\partial}{\partial s}H_k(s), \tag{21}$$

to the input signal $x(\cdot)$ to get $y_{\mathcal{T}k}(\cdot)$ for use in the numerator of (19). The difference between (18) and the one given by [14] is attributed to the different definitions of the STFT. Since the solution to (8) is a linear convolution, the linear phase factor appears in (1). Taken this factor into account, the frequency reassignment expressions become equivalent.

Finally, the kth order recursively reassigned spectrogram can be expressed as in [14]

$$\begin{aligned}y_{\mathcal{R}k}(t,\omega) = \int\int |y_k(t',\omega')|^2 \delta\left[t - \hat{t}(t',\omega')\right] \\ \cdot\delta\left[\omega - \hat{\omega}(t',\omega')\right] dt' d\omega', \end{aligned} \tag{22}$$

where $\delta[\cdot]$ represents the Dirac delta function.

A. Discretization

Using the same discretization method on (20) and (21) yields corresponding digital transfer functions. The relation between the frequency reassignment filter (20) and the original filter (5) is in the discrete case found to be

$$H_{\mathcal{D}k}(z) = \begin{cases} T\sigma^2 \frac{1-\alpha(1-pT)z^{-1}}{(1-\alpha z^{-1})^2}, & \text{if } k = 2; \\ \sigma H_{k-1}(z) + pH_k(z), & \text{if } k \geq 3; \end{cases} \tag{23}$$

Table (II) lists the filter coefficients for orders $k = 2$ to $k = 4$. The relation between the time reassignment filter (21) and the original filter (5) is in the discrete case found to be surprisingly simple, it is given by

$$H_{\mathcal{T}k}(z) = \frac{k}{\sigma}H_{k+1}(z). \tag{24}$$

The discrete reassignment filters are related to three systems of orders $k-1$, k and $k+1$. This means that Table (I), and Table (II) or (23) can be used to find all the needed filter coefficients.

When discretized, the reassignment coordinates (18) and (19) become

$$\hat{f}(n,f) = \mathrm{ROUND}\left(\mathrm{Im}\left[\frac{y_{\mathcal{D}k}(nT, 2\pi f\Omega)}{2\pi\Omega y_k(nT, 2\pi f\Omega)}\right]\right), \tag{25}$$

and

$$\hat{n}(n,f) = n - \mathrm{ROUND}\left(\mathrm{Re}\left[\frac{y_{\mathcal{T}k}(nT, 2\pi f\Omega)}{T y_k(nT, 2\pi f\Omega)}\right]\right), \tag{26}$$

respectively, with $n = 0, 1, \ldots, N-1$ and $f = 0, 1, \ldots, \mathcal{F}-1$, and where Ω denotes the frequency spacing, while $y_{\mathcal{D}k}(\cdot)$ and $y_{\mathcal{T}k}(\cdot)$ represents the sampled outputs after (20) and (21) are

applied to the input signal, respectively. In practice $y_{\mathcal{D}k}(\cdot)$ and $y_{\mathcal{T}k}(\cdot)$ are calculated using (16) with the coefficients found in Table (II) or (23), and (24), respectively. A ROUND$(\cdot)$ function is used to ensure only integer coordinates. It should also be noted that the complex division in (25) and (26) can be avoided by multiplying both numerators and denominators by the complex conjugate of $y_k(\cdot)$.

The discrete kth order recursively reassigned spectrogram can now be expressed as

$$y_{\mathcal{R}k}(nT, 2\pi f\Omega) = \sum_{n'=0}^{N-1}\sum_{f'=0}^{\mathcal{F}-1} |y_k(n'T, 2\pi f'\Omega)|^2 \delta\left[n - \hat{n}(n', f')\right] \cdot\delta\left[f - \hat{f}(n', f')\right], \tag{27}$$

with $n = 0, 1, \ldots, N-1$ and $f = 0, 1, \ldots, \mathcal{F}-1$. Equation (27) means to map and accumulate each point in $|y_k(\cdot)|^2$ onto the new coordinates defined by (25) and (26).

B. Computational Cost, Final notes and Numerical Results

The recursive STFT computed by the presented method is, for fixed f, based on executing (16) $\mathcal{F}$ times. The number of filter coefficients grows linearly with the order k, so when N is the total number of samples in the analyzed signal, $\mathcal{F}$ is the cardinality of the set of chosen frequencies, the cost is $O(k\mathcal{F}N)$. This is also true for the reassigned version, because computing the reassignment coordinates as well just triples $\mathcal{F}N$, and the mapping of the points in (27) is only $O(\mathcal{F}N)$.

The cost can be compared to the classically reassigned STFT spectrogram using the method of Auger and Flandrin [14], [22]. Its cost is either $O(\mathcal{F}N \log N)$ or $O(N\mathcal{F} \log \mathcal{F})$.

The recursive time-frequency reassignment algorithm presented in this paper makes time-frequency reassignment well suited for real-time implementations. The algorithmic complexity is linear in both time and frequency, and the causality of the window allows for online processing. As seen by (16), a cyclic buffer of only $k\Omega$ units is the total required storage space per time instant for the regular version, while the reassigned version needs three times more. Because the reassignment coordinates can point outside the working buffer, this number will in practice be somewhat larger. Also note that the redundancy in the time-dimension can be circumvented by storing a subset of these buffers during processing.

The recursive reassignment method is also seen to eliminate the time-delay introduced by the causal window. The group delay of (5) evaluated on ω can be shown to be k/σ. It is added to the group delay of the input signal when it is passed through the system. Since the partial phase derivative with respect to frequency is the input signal's local group delay around ω, it follows from (19) or (26) that it is subtracted to yield zero delay in the reassigned spectrogram. Thus, the virtually inevitable time-delay of this causal system, is eliminated by the reassignment.

A MATLAB implementation of the regularly and recursively reassigned spectrogram can be found online at `http://www.ii.uib.no/~geirkn/rrspec/`

The synthetic signal used in Figure (1) is a linear chirp signal in the 40 to 80 Hz bandwidth. The sampling rate is $T^{-1} = 1000$ Hz, and the frequency spacing is $\Omega = 0.5$ Hz.

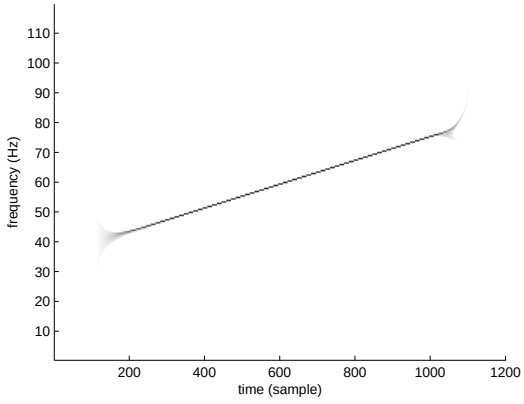

Fig. 1. The 4th order recursively reassigned spectrogram of a linear chirp signal.

IV. Conclusion

A fast algorithm for computing a special case of the reassigned STFT spectrogram has been presented. The corresponding infinite causal STFT window offers a tradeoff between computational cost and time-frequency resolution. The proposed algorithm makes time-frequency reassignment well suited for real-time implementations.

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